

# Classical and quantum N=1 super $W_\infty$ -algebras

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## Abstract

We construct higher-spin N=1 super algebras as extensions of the super Virasoro algebra containing generators for all spins  $s \geq 3/2$ . We find two distinct classical (Poisson) algebras on the phase super space. Our results indicate that only one of them can be consistently quantized.

## 1 Introduction

The infinite-dimensional Virasoro algebra and its extensions play a fundamental rôle in the study of two-dimensional conformal field theories. In particular, the  $W_N$ -algebras [1] are non-linear algebras which contain additional generators, corresponding to fields with conformal spins  $s$  in the interval  $2 \leq s \leq N$ . In contradistinction, the  $W_\infty$ -type algebras [2] [3], generated by an infinite set of higher-spin operators with  $s \geq 1$  or  $2$ , are linear algebras. They appear in the continuum formulation of two-dimensional quantum gravity coupled to  $c = 1$  matter and also in some discrete multi-matrix models which are related to the  $c = 1$  theory [4]-[8]. Our interest in super  $W_\infty$  algebras was raised in a recent paper [9], where we studied the Schwinger-Dyson (S-D) equations of the N=1 supersymmetric eigenvalue model [10], which is a supersymmetric version of the hermitian one-matrix model written in terms of eigenvalues. We found a correspondence between those S-D equations and the bosonic sector of an N=1 super  $W_\infty$ -algebra. In this work, we aim to characterize the full N=1 super algebra, including bosonic and fermionic operators. In fact, we have noticed a lack of explicit formulae in the N=1 case, since the literature mostly concerns N=2 and some of its reductions [11]-[14].

We shall start from a classical realization, that is a Poisson algebra on a phase super space, with a pair of commuting and anti-commuting partners  $(x, \theta)$  and their conjugate momenta,

$(p, \Pi)$  respectively. The “quantum” algebras, announced in the title, will be constructed by replacing momenta by differential operators,  $p \rightarrow -i\hbar\partial/\partial x$  and  $\Pi \rightarrow -i\hbar\partial/\partial\theta$ , and Poisson brackets by commutators. The Planck’s constant  $\hbar$  will be used to control the classical limit ( $\hbar \rightarrow 0$ ) in the usual way,  $\frac{1}{i\hbar}[\cdot, \cdot] \rightarrow \{\cdot, \cdot\}$ . The spin  $s$  of the generators will be classified [14] according to their maximal power in momenta (or derivatives): for the bosonic operators, the maximal power is  $p^{s-1}$ ; for the fermionic ones, we have  $p^{s-1/2}$ . The phase space (or differential) realization is specially suitable for higher-spin extensions, because the Jacobi identity (which is rather cumbersome to check for  $W_\infty$ -algebras) is already built in and it can be effectively used to derive several brackets, so that the calculations become altogether simpler.

In section 2 we describe the classical  $w_\infty$ -algebra, two supersymmetric extensions and a geometric interpretation. The quantization is presented in section 3 and the corresponding classical limit is discussed. Section 4 is dedicated to final comments and conclusions.

## 2 Classical N=1 super $w_\infty$ -algebras

In the bosonic case, the  $w_\infty$ -algebra is equivalent to the Poisson algebra of smooth area-preserving diffeomorphisms on a two-dimensional phase space  $(x, p)$ . Following refs.[2] [3], we introduce the Poisson brackets:

$$\{f(x, p), g(x, p)\} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} . \quad (1)$$

The area-preserving transformations, which preserve the 2-form  $\omega = dx\wedge dp$ , correspond to canonical transformations generated by smooth functions  $\rho(x, p)$  via Poisson brackets,  $f \rightarrow f + \epsilon\{f, \rho(x, p)\}$ . The smooth functions  $\rho$  can be expanded as  $\rho = \sum_{s,n} \rho_{sn} w_n^{(s)}$ , where we take the basis

$$w_n^{(s)} = x^{n+1} p^{s-1} . \quad (2)$$

This set of functions generate the classical  $w_\infty$ -algebra [2]

$$\{w_m^{(r)}, w_n^{(s)}\} = [(s-1)(m+1) - (r-1)(n+1)] w_{m+n}^{(r+s-2)} , \quad (3)$$

which can be seen as a higher-spin extension ( $s \geq 2$ ) of the  $s = 2$  Virasoro algebra generated by  $w_n^{(2)} = x^{n+1} p$ . Introducing a Grassmann-odd spin-3/2 generator  $g_n^{(3/2)}$ , the Virasoro algebra can be extended to a superconformal algebra<sup>1</sup>:

$$\begin{aligned} \{g_m^{(3/2)}, g_n^{(3/2)}\} &= 2w_{m+n+1}^{(2)} , \\ \{g_m^{(3/2)}, w_n^{(2)}\} &= \left[ (m+1) - \frac{1}{2}(n+1) \right] g_{m+n}^{(3/2)} , \\ \{w_m^{(2)}, w_n^{(2)}\} &= (m-n)w_{m+n}^{(2)} . \end{aligned} \quad (4)$$

Assuming the canonical graded Poisson brackets,  $\{x, p\} = 1$ ,  $\{\theta, \Pi\}_+ = -1$ , the most general realization for  $g_n^{(3/2)}$  and  $w_n^{(2)}$ , which is compatible with the infinitesimal conformal transformations

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<sup>1</sup>Throughout this paper, we shall consider the Neveu-Schwarz sector of the superconformal algebra.

$$\delta x = \{x, \epsilon w_n^{(2)} + \alpha g_n^{(3/2)}\} = \epsilon x^{n+1} + \alpha \theta x^{n+1} \quad , \quad (5)$$

$$\delta \theta = \{\theta, \epsilon w_n^{(2)} + \alpha g_n^{(3/2)}\} = \epsilon \frac{(n+1)}{2} x^n \theta + \alpha x^{n+1} \quad , \quad (6)$$

and with the algebra (4), is given by<sup>2</sup>:

$$g_n^{(3/2)}(\lambda) = x^{n+1}(\theta p - \Pi) + 2\lambda(n+1)x^n \theta \quad , \quad (7)$$

$$w_n^{(2)}(\lambda) = x^{n+1}p + (n+1)x^n \left( \lambda + \frac{\theta \Pi}{2} \right) \quad , \quad (8)$$

where  $\lambda$  is an arbitrary real constant.

To include higher-spin generators and extend the super Virasoro algebra (4), we make the following assumptions:

- i) The lowest spin is  $s = 3/2$ .
- ii) There exists a fermionic generator with spin  $s = 5/2$ .
- iii) The Poisson algebra of fermionic generators must obey the rule:

$$\{g^{(r)}, g^{(s)}\} \propto w^{(r+s-1)} + \text{lower spins} \quad .$$

iv) Each generator  $g_n^{(s)}$  is characterized by two indices:  $s$  corresponds to its spin, and  $n$  to its conformal dimension (the eigenvalue of  $L_0 = w_0^{(2)}$ ).

We try the most general Ansatz for the next-spin generator,  $g_n^{(5/2)}$ , in agreement with the assumptions i)-iv), such that the algebra with  $g_n^{(3/2)}$  gets closed:

$$g_{m-1}^{(5/2)} = x^m \theta p^2 + c_m x^m p \Pi + d_m x^{m-1} \Pi + e_m x^{m-2} \theta \quad . \quad (9)$$

In order to calculate the arbitrary constants  $c_m, d_m, e_m$  we verify that:

$$\begin{aligned} \{g_{n-1}^{(3/2)}, g_{m-1}^{(5/2)}\} &= (d_m + 2\lambda n c_m) w_{n+m-2}^{(2)} + (c_m - 1) x^{n+m} p^2 + \\ &+ R_{nm} x^{n+m-1} p \Pi \theta + S_{nm} x^{n+m-2} (\theta \Pi - 2\lambda) + T_{nm} x^{n+m-2} \quad , \end{aligned} \quad (10)$$

where  $R_{nm}, S_{nm}, T_{nm}$  are given functions of  $n, m, c_m, d_m, e_m$ . The next step is to determine the most general linear combinations of the terms on the r.h.s. of (10) (except, of course,  $w_{n+m-2}^{(2)}$  which is already in the algebra), so that they close the algebra with  $g_n^{(3/2)}$ . We define such combinations as:

$$V_m^{(3)} = a_m x^m p^2 + f_m x^{m-1} p \Pi \theta + g_m x^{m-2} (\theta \Pi - 2\lambda) + h_m x^{m-2} \quad . \quad (11)$$

It is easy to see that  $\{g_{n-1}^{(3/2)}, V_m^{(3)}\}$  will produce the term  $x^{n+m-1} \theta p^2$ , among others with lower spins. This term must be part of  $g_{n+m-2}^{(5/2)}$  due to the uniqueness of the spin-5/2 generator.

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<sup>2</sup>Actually, the most general form would be  $g_n^{(3/2)} = x^{n+1}(\theta p - \Pi) + (2\lambda n + \gamma)x^n \theta$ , but the parameter  $\gamma$  can be shifted by a canonical automorphism generated by  $\gamma \ln|x|$ . Therefore, we may take e.g.  $\gamma = 2\lambda$  with no loss of generality.

Indeed, if there were more than one solution for  $c_m, d_m, e_m$  for a given algebra, either the assumption i) or iii) would fail. Therefore, the closure of  $\{g_{n-1}^{(3/2)}, V_m^{(3)}\}$  imposes the following conditions:

$$((m-2n)a_m + f_m) c_{m+n-1} = 2na_m + f_m, \quad (12)$$

$$((m-2n)a_m + f_m) d_{m+n-1} = 2(\lambda n(f_m - 2(n-1)a_m) - g_m), \quad (13)$$

$$((m-2n)a_m + f_m) e_{m+n-1} = (m-2)h_m + 4n\lambda^2(n+m-2)(2(n-1)a_m - f_m). \quad (14)$$

These equations require that  $\lambda = 0$  and we find two possible Ansätze for  $g_n^{(5/2)}$ , corresponding to two different algebras. All higher spins generators,  $w_n^{(s)}$  ( $s \geq 2$ ) and  $g_n^{(s)}$  ( $s > 5/2$ ), are obtained from  $s \leq 5/2$  generators. We end up with  $\lambda = 0$  and two possible N=1 supersymmetric  $w$ -algebras:

**Type 1:** This type is generated by

$$g_n^{(k+3/2)} = x^{n+1} p^k (\theta p - \Pi), \quad (15)$$

$$w_n^{(s)} = x^{n+1} p^{s-1} + \frac{1}{2}(n+1)x^n p^{s-2} \theta \Pi, \quad (16)$$

with the following algebra:

$$\begin{aligned} \{g_m^{(r)}, g_n^{(r')}\} &= 2w_{m+n+1}^{(r+r'-1)}, \\ \{g_m^{(r)}, w_n^{(s)}\} &= [(s-1)(m+1) - (r-1)(n+1)] g_{m+n}^{(r+s-2)}, \\ \{w_m^{(s)}, w_n^{(s')}\} &= [(s'-1)(m+1) - (s-1)(n+1)] w_{m+n}^{(s+s'-2)}. \end{aligned} \quad (17)$$

where  $r, r' = 3/2, 5/2, \dots$  and  $s, s' = 2, 3, 4, \dots$ . In fact, this algebra appeared in [12], in a more complicated realization.

**Type 2:** In this case, the generators split in four families with only even spins in the bosonic sector. The generators are given by:

$$g_n^{(2a+3/2)} = x^{n+1} p^{2a} (\theta p - \Pi), \quad (18)$$

$$\bar{g}_n^{((2a+1)+3/2)} = x^{n+1} p^{2a+1} (\theta p + \Pi), \quad (19)$$

$$w_n^{(2a+2)} = x^{n+1} p^{2a+1} + \frac{1}{2}(n+1)x^n p^{2a} \theta \Pi, \quad (20)$$

$$k_n^{(2a+2)} = x^{n+1} p^{2a+1} \theta \Pi, \quad (21)$$

with the corresponding classical algebra,

$$\begin{aligned} \{g_m^{(r)}, g_n^{(r')}\} &= 2w_{m+n+1}^{(r+r'-1)}, \\ \{g_m^{(r)}, w_n^{(s)}\} &= [(s-1)(m+1) - (r-1)(n+1)] g_{m+n}^{(r+s-2)}, \\ \{w_m^{(s)}, w_n^{(s')}\} &= [(s'-1)(m+1) - (s-1)(n+1)] w_{m+n}^{(s+s'-2)}, \\ \{\bar{g}_m^{(r)}, \bar{g}_n^{(r')}\} &= -2w_{m+n+1}^{(r+r'-1)}, \\ \{\bar{g}_m^{(r)}, w_n^{(s)}\} &= [(s-1)(m+1) - (r-1)(n+1)] \bar{g}_{m+n}^{(r+s-2)}, \end{aligned}$$

$$\begin{aligned}
\{g_m^{(r)}, \bar{g}_n^{(r')}\} &= 2[(r' - 1)(m + 1) - (r - 1)(n + 1)] k_{m+n}^{(r+r'-2)} , \\
\{g_m^{(r)}, k_n^{(s)}\} &= \bar{g}_{m+n+1}^{(r+s-1)} , \\
\{\bar{g}_m^{(r)}, k_n^{(s)}\} &= g_{m+n+1}^{(r+s-1)} , \\
\{k_m^{(s)}, k_n^{(s')}\} &= 0 , \\
\{k_m^{(s)}, w_n^{(s')}\} &= [(s' - 1)(m + 1) - (s - 1)(n + 1)] k_{m+n}^{(s+s'-2)} .
\end{aligned} \tag{22}$$

As far as we know, this algebra has not appeared yet in the literature and we shall call it *super even  $w_\infty$ -algebra*. We note, in passing, that the two algebras (17) and (22) have a sub-algebra in common, generated by  $g_n^{(2a+3/2)}$  and  $w_m^{(2a)}$ . This algebra is called super  $w_{\frac{\infty}{2}}$ , since its bosonic sector corresponds to the  $w_\infty$  truncated to even spins, i.e.  $w_{\frac{\infty}{2}}$ .

Both Poisson algebras are related to area-preserving diffeomorphisms: they preserve the 2-form  $w = dx\Lambda dp - d\Pi\Lambda d\theta$  [11]. The super  $w_\infty$ -algebra corresponds to transformations generated by the following kind of functions:

$$\rho_A = \phi(x + \frac{\theta\Pi}{2p}, p) + (\theta p - \Pi)\psi(x, p) , \tag{23}$$

while the super even algebra is related to generating functions of the form:

$$\rho_B = p\phi(x + \frac{\theta\Pi}{2p}, p^2) + \theta\Pi p\varphi(x, p^2) + (\theta p - \Pi)\psi(x, p^2) + (\theta p + \Pi)p\eta(x, p^2) . \tag{24}$$

Above,  $\phi, \varphi, \psi$  and  $\eta$  are smooth functions of two variables. These generators correspond to two different invariant sub-groups of (super)area-preserving diffeomorphisms. In fact, if  $\rho_1$  and  $\rho_2$  have the form (23), so will have  $\rho_3 = \{\rho_1, \rho_2\}$ . An analogous result holds for functions of the type (24). We recall that, in a general basis, for arbitrary smooth functions  $\rho(x, p, \theta, \Pi)$ , one finds an N=2 super  $w_\infty$ -algebra (see [12] [15]).

### 3 Quantum N=1 super $W_\infty$ -algebra

By “quantum” algebra we mean algebra of commutators, as a quantized version of the Poisson algebras analyzed in section 2. In the bosonic case, the Virasoro algebra is generated by the differential operators

$$L_n \equiv W_n^{(2)} = -i\hbar x^{n+1}\partial , \tag{25}$$

obtained from its classical counterpart  $w_n^{(2)}$  in (2) after the replacement  $p \rightarrow -i\hbar\partial$ . The set of higher spin operators

$$W_n^{(s)} = (-i\hbar)^{s-1} x^{n+1} \partial^{s-1} , \quad s \geq 1 , \tag{26}$$

generate the so called  $W_{1+\infty}$ -algebra, given by:

$$[W_m^{(r)}, W_n^{(s)}] = -i\hbar \sum_{k \geq 0} (-i\hbar)^k C_{mn}^{rs}(k) W_{m+n-k}^{(r+s-2-k)} , \tag{27}$$

$$C_{mn}^{rs}(k) = \frac{1}{(k+1)!} \left( \frac{\Gamma(r)}{\Gamma(r-k-1)} \frac{\Gamma(n+2)}{\Gamma(n-k+1)} - \frac{\Gamma(s)}{\Gamma(s-k-1)} \frac{\Gamma(m+2)}{\Gamma(m-k+1)} \right) . \tag{28}$$

The generator (25) can be generalized<sup>3</sup> into the form (see [13])

$$W_n^{(2)}(\lambda) = -i\hbar \left( x^{n+1} \partial + \lambda(n+1)x^n \right) \quad . \quad (29)$$

We are interested in a basis of operators which satisfy the following condition (originally used to discover the  $W_\infty$ -algebra [3]):

$$[W_m^{(r)}, W_n^{(s)}] = -i\hbar \left( c_0 W_{m+n}^{(r+s-2)} + c_1 W_{m+n-2}^{(r+s-4)} + \dots \right) \quad . \quad (30)$$

This sort of basis is convenient because the algebra can be truncated in only even-spin sub-algebras. Moreover, it admits a central extension [3].

The condition (30) restricts the possible values of the parameter  $\lambda$ . In analogy to the last section, we take an Ansatz for  $W_m^{(3)}$  and we find two solutions (in agreement with [15]):

i) If  $s \geq 1$ , we have  $\lambda = 1/2$  and the  $W_{1+\infty}$ -algebra. The first few generators are given below:

$$\begin{aligned} W_n^{(1)} &= x^{n+1} \quad , \\ W_n^{(2)} &= (-i\hbar) \left( x^{n+1} \partial + \frac{1}{2}(n+1)x^n \right) \quad , \\ W_n^{(3)} &= (-i\hbar)^2 \left( x^{n+1} \partial^2 + (n+1)x^n \partial \right) \quad , \\ W_n^{(4)} &= (-i\hbar)^3 \left( x^{n+1} \partial^3 + \frac{3}{2}(n+1)x^n \partial^2 + \frac{1}{2}n(n+1)x^{n-1} \partial \right) \quad . \end{aligned} \quad (31)$$

Higher spin operators can be obtained via commutators.

ii) If  $s \geq 2$ , one has two equivalent cases,  $\lambda = 0$  or  $1$ . When  $\lambda = 0$  one finds a  $W_\infty$ -algebra, generated by:

$$\begin{aligned} W_n^{(2)} &= (-i\hbar) x^{n+1} \partial \quad , \\ W_n^{(3)} &= (-i\hbar)^2 \left( x^{n+1} \partial^2 + \frac{1}{2}(n+1)x^n \partial \right) \quad , \\ W_n^{(4)} &= (-i\hbar)^3 \left( x^{n+1} \partial^3 + (n+1)x^n \partial^2 \right) \quad , \quad \text{etc.} \end{aligned} \quad (32)$$

The solution  $\lambda = 1$  corresponds to an automorphism of the above generators, leading to an isomorphic  $W_\infty$ -algebra.

Now we present the  $N=1$  supersymmetric extension of the  $W_\infty$ - algebra. First, we introduce an anti-commuting variable  $\theta$  and proceed in analogy to the classical study, by making the following assumptions:

i) The lowest spin ( $s = 3/2$ ) generator [13] [14] is

$$G_n^{(3/2)} = (-i\hbar) \left( x^{n+1} (\theta \partial - \partial_\theta) + 2\lambda(n+1)x^n \theta \right) \quad . \quad (33)$$

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<sup>3</sup>The operators  $L_n(\lambda) = -i\hbar(x^{n+1}\partial + (a + \lambda n)x^n)$  also generate the Virasoro algebra. However, the parameter  $a$  can be arbitrarily shifted by the homeomorphism  $L_n \rightarrow x^{i\hbar c} L_n x^{-i\hbar c} \implies a \rightarrow a + c$ . Thus, we may take  $a = \lambda$ .

Together with the spin-2 operator,

$$W_n^{(2)} = (-i\hbar) \left( x^{n+1} \partial + \frac{1}{2}(n+1)x^n(\theta \partial_\theta + 2\lambda) \right) \quad , \quad (34)$$

they generate the super Virasoro algebra:

$$\begin{aligned} [G_m^{(3/2)}, G_n^{(3/2)}] &= 2i\hbar W_{m+n+1}^{(2)} \quad , \\ [G_m^{(3/2)}, W_n^{(2)}] &= i\hbar \left( (m+1) - \frac{1}{2}(n+1) \right) G_{m+n}^{(3/2)} \quad , \\ [W_m^{(2)}, W_n^{(2)}] &= i\hbar(m-n)W_{m+n}^{(2)} \quad , \end{aligned} \quad (35)$$

whose classical limit coincides with the algebra (4).

ii) We assume the existence of a spin-5/2 generator, whose most general expression is:

$$G_n^{(5/2)} = (-i\hbar)^2 \left( x^{n+1} \partial(\theta \partial + c_n \partial_\theta) + d_n x^n \partial_\theta + e_n x^{n-1} \theta \right) \quad , \quad (36)$$

where the constants  $c_n, d_n, e_n$  must be determined.

iii) The anti-commutation algebra should obey the rule:

$$[G_m^{(r)}, G_n^{(s)}] \propto W_{m+n+1}^{(r+s-1)} + \text{lower spins} \quad . \quad (37)$$

iv) Each operator  $G_n^{(s)}$  is characterized by its spin ( $s$ ) and its conformal dimension ( $n$ ).

Under these assumptions, we find two solutions,  $\lambda = 0$  or  $1/2$ , which are related to each other by an automorphism. Therefore, we may simply take  $\lambda = 0$  and the resulting N=1 super  $W_\infty$ -algebra can be generated by the following basis of operators (we present the lowest spins, since higher spins can be produced by commutators):

$$\begin{aligned} G_n^{(3/2)} &= (-i\hbar) x^{n+1} (\theta \partial - \partial_\theta) \quad , \\ G_n^{(5/2)} &= (-i\hbar)^2 \left( x^{n+1} \partial(\theta \partial + \partial_\theta) + (n+1)x^n \partial_\theta \right) \quad , \\ G_n^{(7/2)} &= (-i\hbar)^3 \left( x^{n+1} \partial^2(\theta \partial - \partial_\theta) - 2(n+1)x^n \partial \partial_\theta - n(n+1)x^{n-1} \partial_\theta \right) \quad , \\ G_n^{(9/2)} &= (-i\hbar)^4 \left( x^{n+1} \partial^3(\theta \partial + \partial_\theta) + 3(n+1)x^n \theta \partial^3 + 3n(n+1)x^{n-1} \theta \partial^2 \right. \\ &\quad \left. + (n-1)n(n+1)x^{n-2} \partial_\theta \right) \quad , \\ W_n^{(2)} &= (-i\hbar) \left( x^{n+1} \partial + \frac{1}{2}(n+1)x^n \theta \partial_\theta \right) \quad , \\ K_n^{(2)} &= (-i\hbar)^2 x^{n+1} \partial \partial_\theta \theta \quad , \\ W_n^{(4)} &= (-i\hbar)^3 \left( x^{n+1} \partial^3 + \frac{3}{2}(n+1)x^n \partial^2 + \frac{1}{2}n(n+1)x^{n-1} \partial \right. \\ &\quad \left. - \frac{1}{2}(n+1)x^n \partial^2 \partial_\theta \theta \right) \quad , \\ K_n^{(4)} &= (-i\hbar)^4 \left( x^{n+1} \partial^3 + (n+1)x^n \partial^2 \right) \partial_\theta \theta \quad , \end{aligned}$$

$$\begin{aligned}
W_n^{(6)} &= (-i\hbar)^5 \left( x^{n+1} \partial^5 + \frac{5}{2}(n+1)x^n \partial^4 + 2n(n+1)x^{n-1} \partial^3 \right. \\
&\quad \left. + \frac{1}{2}(n-1)n(n+1)x^{n-2} \partial^2 (1 + \partial_\theta \theta) - \frac{1}{2}(n+1)x^n \partial^4 \partial_\theta \theta \right) , \\
K_n^{(6)} &= (-i\hbar)^6 \left( x^{n+1} \partial^5 + 2(n+1)x^n \partial^4 + n(n+1)x^{n-1} \partial^3 \right) \partial_\theta \theta .
\end{aligned} \tag{38}$$

We calculated various commutators (up to spin  $s = 6$ ; further commutators can be obtained by means of the Jacobi identity), but we were unable to find a closed form for all structure coefficients. The lowest-spin algebra is listed below:

$$\begin{aligned}
[G_m^{(3/2)}, G_n^{(3/2)}] &= 2i\hbar W_{m+n+1}^{(2)} , \\
[G_m^{(3/2)}, W_n^{(2)}] &= i\hbar \left( (m+1) - \frac{1}{2}(n+1) \right) G_{m+n}^{(3/2)} , \\
[W_m^{(2)}, W_n^{(2)}] &= i\hbar(m-n)W_{m+n}^{(2)} , \\
[G_m^{(5/2)}, G_n^{(3/2)}] &= i\hbar(m-3n-2)K_{m+n}^{(2)} + 2(-i\hbar)^2(m-n)W_{m+n}^{(2)} , \\
[G_m^{(3/2)}, K_n^{(2)}] &= -i\hbar G_{m+n+1}^{(5/2)} + (-i\hbar)^2(n+1)G_{m+n}^{(3/2)} , \\
[K_m^{(2)}, K_n^{(2)}] &= (-i\hbar)^2(n-m)K_{m+n}^{(2)} , \\
[W_m^{(2)}, K_n^{(2)}] &= i\hbar(m-n)K_{m+n}^{(2)} , \\
[G_m^{(5/2)}, W_n^{(2)}] &= i\hbar \left( m - \frac{3}{2}n - \frac{1}{2} \right) G_{m+n}^{(5/2)} + (-i\hbar)^2 n(n+1)G_{m+n-1}^{(3/2)} , \\
[G_m^{(5/2)}, K_n^{(2)}] &= -i\hbar G_{m+n+1}^{(7/2)} + 2(-i\hbar)^2(n+1)G_{m+n}^{(5/2)} + (-i\hbar)^3 n(n+1)G_{m+n-1}^{(3/2)} , \\
[G_m^{(5/2)}, G_n^{(5/2)}] &= -2i\hbar W_{m+n+1}^{(4)} - 3(-i\hbar)^2(n(n+1) + m(m+1))K_{m+n-1}^{(2)} \\
&\quad + 2(-i\hbar)^3((n+m+1)(n+m+2) - 3(n+1)(m+1))W_{m+n-1}^{(2)} , \\
[G_m^{(3/2)}, W_n^{(4)}] &= -i\hbar \frac{1}{2}(n-6m-5)G_{m+n}^{(7/2)} \\
&\quad + (-i\hbar)^2(n(n+1) - 3(m+1)(m+n+1))G_{m+n-1}^{(5/2)} \\
&\quad + (-i\hbar)^3 \frac{1}{2}(m(m+1)(3n+2m+1) + n(n+1)(n-m-2))G_{m+n-2}^{(3/2)} .
\end{aligned} \tag{39}$$

We have also verified that the operators in (38) become the generators (18-21) in the classical limit (given by the associations  $-i\hbar\partial \rightarrow p$ ,  $-i\hbar\partial_\theta \rightarrow \Pi$ , when  $\hbar \rightarrow 0$ ). Therefore, we may say that the generators (38) realize a (quantum) N=1 super  $W_\infty$ -algebra.

Concerning the bosonic sector, composed by  $W_n^{(2s)}$  and  $K_m^{(2r)}$ , it is possible to take linear combinations and find a basis with two decoupled sub-algebras [9]. For instance, if we define

$$\widetilde{W}_n^{(2)} = K_n^{(2)} + i\hbar W_n^{(2)} , \tag{40}$$

the resulting lowest-spin algebra becomes

$$\begin{aligned}
[\widetilde{W}_m^{(2)}, \widetilde{W}_n^{(2)}] &= (-i\hbar)^2(n-m)\widetilde{W}_{m+n}^{(2)} , \\
[\widetilde{W}_m^{(2)}, K_n^{(2)}] &= 0 , \\
[K_m^{(2)}, K_n^{(2)}] &= (-i\hbar)^2(n-m)K_{m+n}^{(2)} .
\end{aligned} \tag{41}$$



This decoupling was also verified for higher spins. The redefined  $\widetilde{W}$ -operators turn out to generate an algebra isomorphic to the even-spin sector of the bosonic  $W_{1+\infty}$ -algebra. On the other hand, the algebra of the operators  $K_n^{(2r)}$  is isomorphic to the even-spin subalgebra of the  $W_\infty$ -algebra<sup>4</sup>. Therefore, the bosonic sector of the super algebra generated by (38) realizes a  $(W_{\frac{\infty}{2}} \oplus W_{\frac{1+\infty}{2}})$ -algebra [9] [13] [14].

It is tempting to call “N=1 super  $(W_{\frac{\infty}{2}} \oplus W_{\frac{1+\infty}{2}})$ -algebra” the one generated by the whole set of operators in (38). We believe this is acceptable at the quantum level, i.e. as long as  $\hbar \neq 0$ . However, in the classical limit ( $\hbar \rightarrow 0$ ) the transformation (40) does not give independent generators and the bosonic sector does not split in two decoupled sub-algebras. This implies that the Poisson algebra generated by (18-21) should not be called a “classical super  $(w_{\frac{\infty}{2}} \oplus w_{\frac{1+\infty}{2}})$ ” – we had better keep the name N=1 super even  $w_\infty$ -algebra.

## 4 Final remarks, conclusion and open questions

We have constructed the N=1 supersymmetric extensions of the  $W_\infty$ -algebras. At the classical level, we found two Poisson algebras, the super  $w_\infty$  and the super even  $w_\infty$ . In the quantum case, we found only one consistent algebra, denominated super  $(W_{\frac{\infty}{2}} \oplus W_{\frac{1+\infty}{2}})$ . Its classical limit coincides with the super even  $w_\infty$ . The algebra  $(W_{\frac{\infty}{2}} \oplus W_{\frac{1+\infty}{2}})$  was first observed in [12] as a truncation of a N=2 super  $W_\infty$ -algebra. We stress that we obtained it in a constructive way, without any embedding in higher algebras.

Although we did not find a general expression for all the quantum operators, we noticed that the available generators can be rewritten in reduced forms. For instance, the bosonic  $K$ -operators in (38) can be expressed as

$$K_n^{(2r)} = (-i\hbar)^{2r} \partial^{r-1} x^{n+1} \partial^r \partial_\theta \theta = p^{r-1} x^{n+1} p^r \Pi \theta \quad . \quad (42)$$

The fermionic operators in (38) can be written as linear combinations of

$$\widetilde{G}_n^{(s+1/2)} = (-i\hbar)^s \left( \partial^{s-1} x^{n+1} \theta \partial + (-)^s x^{n+1} \partial^{s-1} \partial_\theta \right) = p^{s-1} x^{n+1} \theta p + (-)^s x^{n+1} p^{s-1} \Pi \quad . \quad (43)$$

Therefore, we expect the quantum operators to correspond to some special ordering of the classical generators. If we could understand this ordering we might eventually find a closed form for the complete algebra.

We have shown how the super even  $w_\infty$ -algebra can be obtained from the quantum super  $(W_{\frac{\infty}{2}} \oplus W_{\frac{1+\infty}{2}})$  by means of a suitable limit ( $\hbar \rightarrow 0$ ). It is natural to ask whether there is any quantum super  $W_\infty$ -algebra whose classical limit is the super  $w_\infty$  given by eqs.(15-17). We do not have an answer to that question yet. It would also be interesting to study the possible central extensions [3] of these quantum algebras.

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<sup>4</sup>In the quantum case, we may choose a unit system where  $\hbar = 1$ .

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